A Simple Model of Party Formation*

Omer Ali†

July 31, 2013

Abstract

I study a simple, three player model of legislative bargaining over ideological and distributive dimensions. When preferences are quasi-linear, there exists a symmetric equilibrium, in which the median legislator randomizes between proposing to either extreme legislator, and where the extreme legislators always propose to the median. I also show that there exist asymmetric equilibria for values of the ideal policy of the median legislator in the neighborhood of $\frac{1}{2}$. Finally, I define and characterize stable parties.

*I would like to thank Ichiro Obara and Jernej Copic for encouragement and many helpful comments and suggestions. I would also like to thank Joe Ostroy, Amrita Dhillon and participants at the UCLA Theory Proseminar for helpful comments and suggestions. Remaining errors are the sole responsibility of the author.

†omer.ali@duke.edu; Samuel DuBois Cook Center, Duke University

1 Introduction

There is, as yet, no consensus on what the most important functions of political parties are (Dhillon (2003)). Baron (1993) focuses on parties' wish to influence the ultimate policy that the government implements, as well as their incentives to garner wide electoral support. In his model, parties consider the eventual government formation process as well as the process of bargaining between parties into account when choosing their policies. They also take into account the potential electoral support that their policies are likely to attract. The model has two main results: in a 3-party set-up with a two dimensional policy space, either all parties choose the median policy, or else, for some parameter values, they each choose different positions that are equidistant and symmetric from the median position. Although the title of the paper[1] alludes to the process through which parties form, the main contribution of the paper is its attempt to consider multiple goals that parties have simultaneously.

Baron and Ferejohn (1989) is the first study to model the process of legislative bargaining. They modeled the bargaining process as a dynamic game between agents bargaining over the split of a finite resource. A proposer is chosen with some exogenous probability at the beginning of each period, and they propose a split of the resources that must garner majority support in order to be implemented. The equilibrium strategies that emerge are ones in which the proposer courts a minimal winning coalition that approves the proposal, while the other legislators oppose it. There is no role for political parties in this model. As a result, every proposal is passed with a minimal winning coalition and legislators vote independently of any coalitional considerations.

Jackson and Moselle (2002) extend the basic bargaining model in Baron and Ferejohn by adding a policy that the proposer can choose. This choice is characterized as a location in the interval $[0, 1]$. In this setting, parties are exogenous constraints on individual legislators ensuring that they propose a particular vector of policy and allocations whenever they are selected. In a 3 legislator example (left, right and media legislators), the authors show that *stable parties*[2] emerge only between players that are ideologically adjacent - a party between the two extreme legislators, for example, is not stable.

[1]Government formation and endogenous parties.

[2]A political party is defined to be stable if no agent wishes to leave in anticipation of higher utility elsewhere.

Morelli (2004) studies a three party model, where party formation is modeled as two of the three parties agreeing to a coalition. In this setting, the parties serve two functions. They are a commitment device for politicians who want to run on platforms that are not their own. The second function they serve is as a coordination device for voters. Eguia (2007) on the other hand models parties as coalitions that vote according to the wishes of the majority of their members. A peculiar assumption in the model is that legislators face uncertainty over their own preferences. At the beginning of each stage, legislators form parties prior to the resolution of uncertainty, and credibly commit to vote according to the party line once their preferences are revealed. In a model with repeated stages of voting, Eguia (2012) shows that a stable coalition of legislators (a party) can consistently shift equilibrium policy away from the status quo median - this conclusion contradicts one of the results in Baron (1993), which argues that under some circumstances, parties will simply offer the median policy. We follow the approach of Jackson and Moselle (2002) and adopt their framework. In their study, they show that an equilibrium in stationary strategies exists. In this paper, we show that a symmetric equilibrium exists in a legislative game with 3 players, and that the equilibrium takes a particular form. Moreover, we show that an equilibrium exists in some neighbourhood of this symmetric equilibrium. Finally, we demonstrate the properties of equilibrium quantities with numerical simulations. Section 2 describes the model, section 3 discusses existence and properties of equilibrium, section 4 explores the stability of political parties, and section 5 concludes.

2 The model

2.1 Legislative game

Consider the legislative bargaining model in Jackson and Moselle (2002) with three players. Let the ideal point of legislator $i \in \{0, m, 1\}$ be $v_i \in [0, 1]$, and let $v_0 = 0$ and $v_1 = 1$ and m be the median legislator. A legislator is picked proposer with probability p_i (this will be assumed to be 1/3 for the remainder of the paper) at the beginning of each period to propose a *decision* composed of a policy y^i and a split of the finite resources available, X: $\{x_i\}_i$ such that $\sum_i x_i = 1$. A decision is then voted on by all players who can specify if they vote "for" or "against" the decision.

It is adopted if it receives majority support. There are an infinite number of periods denoted by $t \in \{1, 2, \ldots\}$. The game ends when a decision is adopted. A strategy for each player is a decision that they propose whenever they are chosen proposer, and a voting rule whenever they are not proposing that can be characetrized by a function assigning a "yes" or "no" to each possible decision . A strategy for a player is *stationary* if their continuation strategy at the beginning of every subgame is the same regardless of history. An equilibrium is *stationary* if it is a subgame perfect equilibrium and each legislator's strategy is stationary.

A *simple equilibrium* is a stationary equilibrium in which (i) each legislator randomizes over at most $M < \infty$ proposals, (ii) each such proposal can be identified with a distinct coalition C such that $i \notin C$, and $\#C = \frac{n-1}{2}$. In our version of the model with only three legislators, each C is simply another player.

2.2 Preferences

Let legislator i evaluate a decision $d = (y, x) = (y, x_0, x_m, x_1)$ with utility function u_i, defined as follows:

$$u_i(d) = x_i X + 1 - (v_i - y)^2$$

This function is separable in x_i and y, is concave in the decision, d, and is single peaked for every x for legislator i in y at their ideal point, v_i. Therefore, it satisfies the assumptions in Jackson and Moselle (2002). It is commonly assumed that legislators are office-seeking: they only care about getting re-elected. This form of utility function can be interpreted in that light if we consider the ideal point v_i to be the ideal point of the legislator's district, and the resources $x_i X$ as the resources that the legislator then spends on their district (or redistributes among voters). In either case, we can view the legislator as having internalized the preferences of their district. Since pursuing the best interest of the district (or the district's median voter) is the best way to ensure reelection, this formulation is in line with the assumption that legislators are office-seeking. Alternatively, if we assume that legislators are policy-motivated, then v_i can be interpreted as their own ideal point, and they maximize their own utility.

3 Equilibrium

In this section, we show the existence of a novel kind of equilibrium in the three player game. One in which the median player randomizes between proposing to the other two players, and where the extreme players always propose to the median, when either one is chosen proposer. This equilibrium can be seen as exhibiting some form of party structure since legislators only propose to adjacent legislators - any proposal that passes does so with a contiguous coalition. In the next section, we explore an explicit party structure using the notion of the stability of coalitions.

Proposition 2[3] in Jackson and Moselle (2002) shows that a simple equilibrium exists in the general legislative game with unspecified concave utlity functions and n legislators. Here, we will show that a particular kind of simple equilibrium exists. First, we show that a symmetric simple equilibrium exists wheneve the median legislator is equidistant from the extreme legislators. By Proposition 3 in Jackson and Moselle, any approved decision in any stationary equilibrium distributes resources among an exact majority. This means that each proposer targets another player, and offers that player a share of the resources that would induce them to vote for the proposal. In an arbitrary symmetric equilibrium, it may be the case that legislators randomize between proposing to one of the other two players. We show, however, that whenever a symmetric equilibrium exists, it is one where only the median player randomizes (with probability 1/2) between proposing to 0 and 1, and where the other two players always propose to the median player.

Claim 1. *A symmetric equilibrium exists whenever $v_m = 1/2$ in which:*

- *m proposes $(1 - x^m, 1/4)$ to 0 with probability 1/2 and $(1 - x^m, 3/4)$ to 1 with probability 1/2*
- *0 proposes $(1 - x^0, 1/4)$ to m whenever they are chosen proposer*
- *1 proposes $(1 - x^0, 3/4)$ to m whenever they are chosen proposer*
- *players that are proposed to always accept*

Having shown the existence of a symmetric equilibrium, we will now show that it must be one where both extreme players always propose to the median player, and

[3] If u_i is concave for each i, then there exists a simple equilibrium. Moreover, if each u_i is strictly concave then all stationary equilibria are simple (Jackson and Moselle (2002)).

the median player randomizes with probability 1/2 between proposing to either one. The proof will consider two cases: first, we will consider the case of a symmetric equilibrium in which the two extreme players propose to one another with certainty and show that, for both players, proposing to the median player instead is a profitable deviation. Second, we will consider a symmetric equilibrium in which the two extreme players randomize between proposing to the other extreme player and proposing to the median. We will then show that it is profitable for both players to propose to the median player every time, and so randomizing is not equilibrium behaviour.

It is important to point out that a crucial assumption throughout this paper is that the amount of resources, X, is sufficiently large to allow for an interior solution (where the proportion of resources shared with the legislator being proposed to is strictly between 0 and 1). This allows the proposed policy to always be at the midpoint between the two legislators' (the proposer and the legislator accepting the proposal) ideal points.

The first two results we prove show that the equilibrium in which the median player randomizes between proposing to 0 and 1, and where both 0 and 1 propose to the median with certainty is the unique symmetric equilibrium in the three player game specified above.

Claim 2. *Let $v_m = 1/2$. For some values of δ and large enough values of X, a symmetric equilibrium in which 0 and 1 always propose to m, and m randomizes with probability 1/2 between proposing to either extreme player exists. Moreover, it is the unique symmetric equilibrium.*

Next, we show that, in addition to a symmetric equilibrium, whenever $v_m = 1/2$, there exist equilibria when the position of the median legislator is near v_m. These equilibria will not be symmetric and some of them are described in the numerical analysis we conduct in the appendix.

Claim 3. *An equilibrium in which 0 and 1 always propose to m, and m randomizes between proposing to either player exists for v_m in the neighbourhood of 1/2.*

The appendix contains numerical simulations of equilibrium quantities and demonstrates how they change with parameters δ and X. We can see that the range of δ in which an equilibrium exists is increasing in X: the larger X is, the larger the range of δ at which an equilibrium exists (appendix sections 5.1, 6.1, 6.2). Moreover, we can see that at any value of X, the proportion kept by the median player whenever they propose to the extreme players is increasing in δ (appendix section 6.4).

4 Political Parties

Jackson and Moselle define a *party* as a subset of legislators $P \subset N$ who propose the same policy whenever any member is chosen proposer. Moreover, they each vote for this proposal whenever it is tabled. Such a party is *stable*[4] if the following condition holds:

$$u_i(P) \geq u_i(P') \ \forall i \in P, \ \forall P' \ni i$$

In their examples, Jackson and Moselle find that the identity of stable parties depends on the specific utility parameters. Although our framework specifies the utility function, it is parameter-free[5]. To this extent, there is an obvious question to ask: what is the identity of stable parties in our framework? How do they differ with the location of the median legislator? Is it possible to have a non-contiguous stable party (one made up of legislators 0 and 1)?

Notice that a party generates some joint utility for its members:

$$u^P = u_i(P) + u_j(P) = X + u(v_i - v^P) + u(v_j - v^P),$$

where v^P is the policy proposed by party members. For $u_i(P)$ to be well defined, the split of this total utility among the members of the party must be specified. Since there are always two players in a party in our example, and following Jackson and Moselle, we let the split of utility be determined by the Nash bargaining solution[6].

[4] Although the notion of stability defined here applies more generally than the three player model we describe, the utility that an individual player obtains from being a member of a party is not well defined for more than three players. We use the Nash bargaining solution to determine the policy that a party chooses and split the revenue between two legislators in a party. However, in a party with more than 3 players, this approach is no longer applicable. The definition of stability above, therefore, applies in those cases, provided utility is defined in some way.

[5] By this, we mean that, as opposed to Jackson and Moselle who must specify a parameter that determines, for each legislator, his relative preference for policy vis-a-vis resources, we have a utility function that is common for all legislators. One possible variation to explore may be the concavity of the utility function, specified by the power to which we raise the distance between policy and ideal point.

[6] The agreement $x^* \in X$, the set of possible consequences, is a Nash solution of the bargaining

Let $P(i,j)$ be the party formed by players i and j. We begin with the equilibrium described above (where both extreme players propose to m, and m randmoizes) and investigate which parties are stable. First, consider the case of $P(0,m)$ and let v^p_{0m} be the chosen party policy and x^0_{0m} be the proportion of the resources kept by player 0 in the split. We're looking for the Nash bargaining solution, and so (v^p_{0m}, x^0_{0m}) solve the following maximization:

$$(v^p_{0m}, x^0_{0m}) = \arg\max_{v,x} \left\{ (xX + 1 - v^2 - \bar{u}^0)((1-x)X + 1 - (v_m - v)^2 - \bar{u}^m) \right\},$$

where $\bar{u}^i$ is expected utility of player i in the equilibrium described above. Since we're considering the symmetric case, $v_m = 1/2$. The first order conditions are then simply:

$$FOC(x): \; x = \frac{1}{2X}\left(X - (v_m - v)^2 - \bar{u}^m + v^2 + \bar{u}^0\right)$$

$$FOC(v): (v_m - v)\left(xX + 1 - v^2 - \bar{u}^0\right) - v\left((1-x)X + 1 - (v_m - v)^2 - \bar{u}^m\right) = 0$$

Solving these two simultaneous equations, we get $v^P_{0m} = v_m/2$ and $x^0_{0m} = \frac{X+\bar{u}^0-\bar{u}^m}{2X}$. We can repeat this exercise for the party $P(0,1)$ and find $v^P_{01} = 1/2$ and $x^0_{01} = \frac{X+\bar{u}^0-\bar{u}^1}{2X}$. Finally, for the party $P(m,1)$, the quantities are $v^P_{m1} = (1+v_m)/2$ and $x^m_{m1} = \frac{X+\bar{u}^m-\bar{u}^1}{2X}$. These results are summarized in the following proposition.

Proposition 1. *Let X be sufficiently large, and let $P(i,j)$ be a party composed of players i and j. If $\bar{u}^k$ is player k's expected utility from the independent voting game and the players determine the policy of the party, v^P_{ij}, and the share of resources, x^k_{ij}, allocated to player k through the Nash bargaining solution, then these quantities take the following form:*

$$v^P_{ij} = \frac{v_i + v_j}{2} \qquad x^i_{ij} = \frac{X + \bar{u}^i - \bar{u}^j}{2X}$$

Notice that the party's policy is the midpoint between the two legislators' ideal points, and the share of resources allocated to a legislator is proportional to their expected utility in the independent voting game and decreasing in the other player's expected utility in the independent voting game. This makes intuitive sense since

problem $\{X, (d_1, d_2), \succsim_1, \succsim_2\}$ if and only if:

$$(u_1(x^*) - d_1)(u_2(x^*) - d_2) \geq (u_1(x) - d_1)(u_2(x) - d_2) \forall x \in X,$$

where (d_1, d_2) is the disagreement point, $\succsim_i$ is the preference relation of i over lotteries over X (page 302, Osborne & Rubinstein (1994)).

more influential players (those with higher expected utility in the independent voting game) command a larger proportion of the party's resources.

According to the above definition of stability, a party is stable if neither member wishes to be part of another party. Let's consider the party $P(0,m)$. 0 does not want to leave the party whenever $U^0(P(0,m)) \geq U^0(P(0,1))$, while m doesn't want to leave whenever $U^m(P(0,m)) \geq U^m(P(m,1))$. Below, we derive the general conditions for stability of $P(0,m)$ and proceed to show that these hold in the symmetric equilibrium we discuss above.

Notice that since the threat points are the expected utilities from the independent voting game, it must be the case that $\bar{u}^0 = \bar{u}^1$, otherwise m would not randomize between propositing to the extreme players. This observation makes it clear to see that $U^m(P(0,m)) \geq U^m(P(m,1)) \implies v_m \leq 1/2$, the first condition for stability of $P(0,m)$.

The second condition for stability comes from the condition for player 0 preferring not to leave the party and it simplifies to[7]:

$$\frac{X + \bar{u}^0 - \bar{u}^m - 1}{2} + \frac{1 - v_m^2}{4} \geq 0$$

Consider the symmetric case ($v_m = 1/2$, and m randomizes with probability 1/2 between the two players) and notice that the first condition is weakly satisfied. To check the second condition, we need to recall the expressions for $\bar{u}^0$ and $\bar{u}^m$. We reproduce these below and, since we are looking at the symmetric case, simplify the expressions[8] so that $x_1^m = x_0^m = x^m$ and $x_m^0 = x_m^1 = x^0$.

$$\bar{u}^0 = \frac{1}{3}\left[x^0 X + 1 - (1/4)^2 + 1 - (3/4)^2 + (1/2)\left((1 - x^m)X + 1 - (1/4)^2\right) + (1/2)(1 - (1/4)^2)\right]$$

[7]It's straight forward to derive this inequality from the party policies and proportion of party resources allocated to each member in parties $P(0,m)$ and $P(0,1)$ and the following definition:

$$u^i(P(i,j)) = x_{ij}^i X + 1 - (v_i - v_{ij}^P)$$

[8]Namely, the proportions offered by m to both 0 and 1 are the same and the proportions offered by both 0 and 1 to m are the same. We chose to denote these with a 0 superscript for notational convenience, and a 1 subscript could have also been used.

$$\bar{u}^m = \frac{1}{3}\left[(1/2)(x^m X + 1 - (1/4)^2) + (1/2)(x^m X + 1 - (1/4)^2) + \left((1-x^0)X + 1 - (1/4)^2\right) \right. \\ \left. + \left((1-x^0)X + 1 - (1/4)^2\right)\right]$$

A straight forward substitution and simplification shows that the second condition for stability of $P(0,m)$ can be expressed as follows:

$$\frac{1}{2}\left(X + \frac{1}{3}\left(X(3x^0 - (3/2)x^m - (3/2)) + (1/4)^2 - (3/4)^2\right) - 1\right) + \frac{1-(1/2)^2}{4} \geq 0$$

We will show that this inequality holds for large values of X. To do this, notice that we only need to show that the coefficient on X is positive, in which case, as X increases the left hand side increases and eventually becomes positive, since the negative terms are constant. The coefficient on X can be expressed as follows:

$$\frac{1}{2} \times \frac{1}{3}\left(3 + 3x^0 - \frac{3}{2}x^m - \frac{3}{2}\right) = \frac{1}{2}\left(1 + x^0 - \frac{1+x^m}{2}\right) > 0$$

The inequality follows from the fact that $x^m \leq 1$. This result is summarized in the following proposition.

Proposition 2. *In a symmetric game with a sufficiently large X, parties $P(0,m)$ and $P(m,1)$ are stable, while $P(0,1)$ is not.*

Proof. We've already shown the stability of $P(0,m)$ above. The stability of $P(m,1)$ follows from the symmetry of the set up. $P(0,1)$ is unstable since both 0 and 1 prefer to be in a party with m. □

5 Conclusion

In this paper, we showed that a particular kind of symmetric equilibrium exists in a 3-legislator version of Jackson and Moselle's legislative bargaining model with quasi-linear utilities. This is a first step to studying political party formation in this framework. Although Jackson and Moselle set out to study political party formation, they only study this under restrictive assumptions on the utility function[9]. We also characterize the set of stable parties.

[9] They look at linear utility functions with a coefficient that determines how much a legislator values the proximity of a policy to their ideal point relative to consumption of resources.

There are some outstanding questions. The first is whether we can show the existence result in claim 2 analytically. We conjecture that this is indeed possible. Second, it remains to be seen whether claim 3 can be shown more completely (as it stands, the final step of the proof is assumed to hold).

The results shown here apply to a restricted model: there are only three legislators with a particular utility function. The first question we might ask is whether these results hold for any quasi-linear utility function. This is quite possibly true. Whether or not the results hold for more than three players, however, is a less straightforward question. Most importantly, the utility of a legislator from belonging to a party is only well defined for parties consisting of two legislators since the internal split of the resources is decided by the Nash bargaining solution. With parties that have more than just two members, this is no longer possible. How would the internal split of resources be decided in this case? This is an open question.

Furthermore, we only considered the symmetric case. However, it would be interesting to explore how the identity of stable parties, for example, changes a the position of the median legislator changes. In a model with more than three players, one questions that would be interesting to explore is whether parties are always composed of contiguous coalitions of legislators.

It may be the case that the current framework would not lend itself well to exploring these questions, particularly those that extend the model to more than 3 legislators. As we can see, the model quickly becomes unwieldy and to obtain any properties of the model usually requires solving some set of simultaneous equations. With more legislators, this set would increase, making us resort even more frequently to numerical anaysis, instead of analytical results.

6 References

Baron, David (1993), “Government Formation and Endogenous Parties”, The American Political Science Review, Vol. 87, No. 1 (Mar., 1993), pp. 34-47

Baron, David and John Ferejohn (1989), “Bargaining in legislatures”, The American Political Science Review, Vol. 83, No. 4 (Dec., 1989), pp. 1181-1206

Dhillon, Amrita (2003), “Political Parties and Coalition Formation,” Warwick Economic Research Papers 697

Eguia, Jon (2012), “A spatial theory of party formation”, Economic Theory, Springer, vol. 49(3), pages 549-570, April.

Jackson, Matthew O. and Boaz Moselle (2002), “Coalition and Party Formation in a Legislative Voting Game,” Journal of Economic Theory, Elsevier, vol. 103(1), pages 49-87, March.

Morelli, Massimo (2004) “Party Formation and Policy Outcomes under Different Electoral Systems,” Review of Economic Studies, Wiley Blackwell, vol. 71, pages 829-853, 07.

Osbore, Martin J. and Ariel Rubinstein (1994), “A Course in Game Theory”, MIT Press, Cambridge, Massachusetts

7 Appendix

7.1 Proofs

7.1.1 Proof of Claim 1

Proof. A symmetric equilibrium is a vector

$$(q_m^0, x_m^0, q_1^0, p, x_0^m, x_1^m, q_m^1, x_m^1, x_0^1)$$

such that $q = q_m^0, = q_m^1$, $x_m = x_m^0 = x_m^1$, $x = x_1^0 = x_0^1$, $x_0^m = x_1^m = x^m$, and $p = 1/2$. Namely, the probailities with which 0 and 1 randomize between proposing to one another and proposing to m are the same, and the amount of resources they offer is the same. In addition, the amount of resources offered by m to either 0 or 1 is the same, and m randomizes between offering to 0 and 1 with equal probabilities. A symmetric equilibrium, therefore, can be characterized by 4 quantities:

$$(q, x_m, x, x^m)$$

Define $A_i^s(l) := \{d \in D : u_i(d) \geq \delta v_i(l)\}$, where $l \in L^s$, the space of symmetric strategy profiles. We want to show that $A_i^s(l)$ is (i) non-empty, (ii) compact, and (iii) continuous.

(i) Non-empty: Since $A_i^s(l)$ is non-empty for $l \in L$ (Jackson and Moselle (2002)), and $L^s \subset L$, it follows that $A_i^s(l)$ is non-empty.

(ii) Compact: $A_i^s(l)$ is closed, and since $A_i^s(l) \subset [0,1]^6$, which is compact, it follows that $A_i^s(l)$ is compact.

(iii) Continuous: We want to show that $A_i^s(l)$ is both uhc and lhc:

uhc: A correspondence $f : X \to Y$ is uhc if $\forall$ compact $B \subset X$, the set $f(B) = \{y \in Y : y \in f(x)\ for\ some\ x \in B\}$ is bounded.
Since D^s is compact, $A^s(B) \subset D^s$ is compact for any compact B, so $A_i^s(.)$ is uhc.

lhc: A correspondence $f : X \to Y$ is lhc if $\forall x^n \to x \in X : x^n \in X \forall n$, $\forall y \in f(x)$, $\exists y^n \to y$ and $N \in \mathbb{N} : y^n \in f(x^n) \forall n > N$
Let $l^n \to l \in L^s$, $l^n \in L^s\ \forall n$. Pick $d \in A_i^s(l) = \{d \in D : u_i(d) \geq \delta v_i(l)\}$.

We want $d^n \to d$ and N such that $d^n \in A_i^s(l^n)\ \forall n > N$. Let d_i be the best decision for player i - they get all the resources ($x_i = 1$), and the proposed policy is their ideal point ($y = v_i$). It's clear that the following inequality holds:

$$u_i(d_i) > \delta v_i(l^n)$$

Let $d^k = \alpha_k d_i + (1-\alpha_k)d$, where $\alpha_k \to 0$ (i.e. $d^k \to d$). For each l^n, we can find a k_n such that $u_i(d^{k_n})) > \delta v_i(l^n)$. Then $d^{k_n} \to d$ and $d^{k_n} \in A_i^s(l^n)\ \forall n$, hence A_i^s is lhc.

Define $A_{-i}^s(l) = \cup_{j\neq i} A_j^s(l)$ - the set of decisions that would be approved if proposed by player i. We want to show that $A_{-i}^s(l)$ is (i) nonempty, (ii) compact, and (iii) continuous.

(i) Non-empty: Since each $A_j^s(l)$ is non-empty, so is $A_{-i}^s(l)$.

(ii) Compact: Since a finite union of compact sets is compact, $A_{-i}^s(l)$ is compact.

(iii) Continuous: The union of uhc (lhc) correspondences is uhc (lhc). The footnote contains a direct proof [10].

Define $A_i^*(l) = \underset{d\in A_{-i}}{\arg\max}\{u_i(d)\}$. By the maximum theorem, $A_i^*(l)$ is non-empty, compact-valued, and uhc.

[10] We want to show that $A_i^s(l)$ is both uhc and lhc:

uhc: Let $B \subset L^s$ be a compact set. Since $A_{-i}^s(B) = \cup_{j\neq i} A_j^s(B)$ and each $A_j^s(B)$ is bounded, so is $A_{-i}^s(B)$.

lhc: Suppose we can express l as $(\ldots, d_{iC}, \pi_{iC}, \ldots)$, where π_{iC} is the probability with which player i proposes d_{iC} to coalition C. Define $E[l]$ in the following way:

$$E[l] = \sum_i p_i \left(\sum_C \pi_{iC} d_{iC} \right)$$

Since individuals are risk averse,

$$u_i(E[l]) > \delta v_i(l)\ \forall i$$

To see this, notice that $v_i(l) = \sum_j p_j \left(\sum_C \pi_{jC} u_i(d_{jC})\right)$.
Let $d^k = \alpha_k E[l] + (1-\alpha_k)d : \alpha_k \to 0$. For each $j \neq i$,
For each $j \neq i$, $\exists K_j : u_j(d^k) > \delta v_i(l^n)\ \forall k \geq K_j$

Let $N_n = \max\{K_j, K_k\}$. Then $u_j(d^k) > \delta v_j(l)\ \forall k \geq N_n$, Hence, $d^{N_n} \in A_{-i}^s(l)$. Since $d^{N_n} \to d$ and $d^{N_n} \in A_{-i}^s(l^n)$, $A_i^s(l)$ is lhc.

Define $H^s(l) = \{\tilde{l} \in L^s : \tilde{d}_{iC} \in A_i^*(l)\ \forall i\}$

We want to show that $H^s(l)$ is (i) non-empty, (ii) compact, (iii) uhc, and (iv) convex.

(i) Non-empty: Each $A_i^*(l)$ is non-empty. To show that $\tilde{l}$ is symmetric, notice that if $d_{01} \in A_0^*(l)$, then, by symmetry, it must be that $d'_{01} \in A_1^*(l)$ (where d'_{01} is the proposal in which 1 offers to 0 what 0 offers to 1 in d_{01}).

(ii) Compact: Since $H^s(l) \subset L^s$, a compact space, it suffices to show that $H^s(l)$ is closed. Let $l^n \to l$, where $l^n \in H^s(l')\ \forall n$. Then it follows that $d_{iC}^n \in A_i^*(l')\ \forall i$. Let $l^n = (l_0^n, l_m^n, l_1^n)$, where l_i^n is the continuation strategy of player i - $l_i^n = (\ldots, (d_{iC}^n, \pi_{iC}^n), \ldots)$. If $l^n \to l$, it must be the case that $l_i^n \to l_i\ \forall i$ and hence $(d_{iC}^n, \pi_{iC}^n) \to (d_{iC}, \pi_{iC})$.
$l^n \in H^s(l') \Longrightarrow (\pi_{iC}^n > 0 \Rightarrow d_{iC}^n \in A_i^*(l'))\ \forall i$. This means that $u_i(d_{iC}^n) = \bar{u}_i \geq u_i(d)\ \forall d \in A_{-i}^s(l')\ \forall n$. Since $u_i(.)$ is continuous, $u_i(d_{iC}) = \bar{u}_i \Longrightarrow d_{iC} \in A_i^*(l')$. Now, to show that $l \in H^s(l')$, we simply need to show that $\pi_{iC} > 0 \Longrightarrow d_{iC} \in A_i^*(l')$. Suppose $\pi_{iC} > 0$, then it must have been the case that $\pi_{iC}^n > 0$ for $n \geq N$, for some large N. But if this is true, then we have shown that it must be the case that $d_{iC} \in A_i^*(l')$.

(iii) Uhc: Since L^s is bounded, it follows that $H^s(B) \subset L^s$ is bounded for any compact B, therefore $H^s(.)$ is uhc.

(iv) Convex: Suppose $l, l' \in H^s(l'')$ and let $\lambda \in (0,1)$, then if $l_\lambda = \lambda l + (1-\lambda)l'$, $l_\lambda \in H^s(l'')$. To see this, notice that we simply need to show that l_λ is a symmetric strategy profile. Recall that the entries of a symmetric continuation strategy are decisions and probabilities:

$$((d_{01}, 1-q), (d_{0m}, q), (d_{m0}, 1/2), (d_{m1}, 1/2), (d_{1m}, q), (d_{10}, 1-q)),$$

where each proposal d_{iC} is identified with a distinct C (one of the other players). The difference between l and l' are the probability, q, and the details of d_{iC}, by which we mean the particular proportions being offered to C. Since each $A_j^s(l'')$ is convex, if both d'_{iC} and d_{iC} maximize u_i over $A_j^s(l'')$, then, by the convexity of $A_j^s(l'')$ and concavity of $u_i(.)$, it must be that d_{iC}^λ also maximizes $u_i(.)$. This means that $d_{iC}^\lambda = \lambda d_{iC} + (1-\lambda)d'_{iC} \in A_i^*(l'')$. If both l and l' are symmetric (and hence have the form above), then l^λ is also symmetric.Then if l^λ is symmetric and each d^λ_{iC} maximizes u_i over $A^*_i(l'')$, it follows that $l^\lambda \in H^s(l'')$.

By Kakutani's fixed point theorem, $H^s : L^s \to L^s$ has a fixed point. This fixed point is a symmetric equilibrium of the legislative game[11]. □

7.1.2 Proof of Claim 2

Proof. First, notice that when X is large enough, the optimal policy y^i_j for player i to propose to player j is $y^i_j = \frac{v_i+v_j}{2}$, which is the midpoint between the ideal points of i and j. To see why this is the case, let x^i_j be the proportion that i keeps when they propose to j and notice that whenever player i chooses what policy to propose, they solve the following maximization problem:

$$\max_{y^i_j, x^i_j} \left\{ U_i(y^i_j, x^i_j) : U_j(y^i_j, x^i_j) \geq \delta v_j \right\},$$

where v_j is j's continuation payoff - a function of all players' strategies.

Since $U_i(y,x) = u_i(y) + x^i_j X$, and $v_j = \frac{\delta}{3}\left(U_j(P_j) + u_j(y^i_j) + x^i_j + U_j(P_k)\right)$, the lagrangian of the above maximization problem is the following:

$$\mathcal{L} = u_i(y) + x^i_j X + \lambda \left(u_j(y) + (1-x^i_j)X - \frac{\delta}{3}\left(U_j(P_j) + u_j(y) + (1-x^i_j)X + U_j(P_k)\right) \right)$$

The first order conditions of the above maximization problem with respect to y and x^i_j are the following condition:

$$FOC(y) : \frac{\partial u_i(y)}{\partial y} + \lambda \left(\frac{\partial u_j(y)}{\partial y} - \frac{\delta}{3}\frac{\partial u_j(y)}{\partial y} \right) = \frac{\partial u_i(y)}{\partial y} + \lambda \frac{\partial u_j(y)}{\partial y}\frac{3-\delta}{3} = 0$$

$$FOC(x^i_j) : X + \lambda \left(-X + \frac{\delta}{3} X \right) = X \left(1 - \lambda \frac{3-\delta}{3} \right) = 0$$

The first order condition with respect to x^i_j yields $\lambda = \frac{3}{3-\delta}$. Substituting this into the expression for the first order condition with respect to y yields:

$$\frac{\partial u_i(y)}{\partial y} + \frac{\partial u_j(y)}{\partial y} = 0$$

[11] The final claim follows from the final step of the existence proof in Jackson and Moselle (2002)

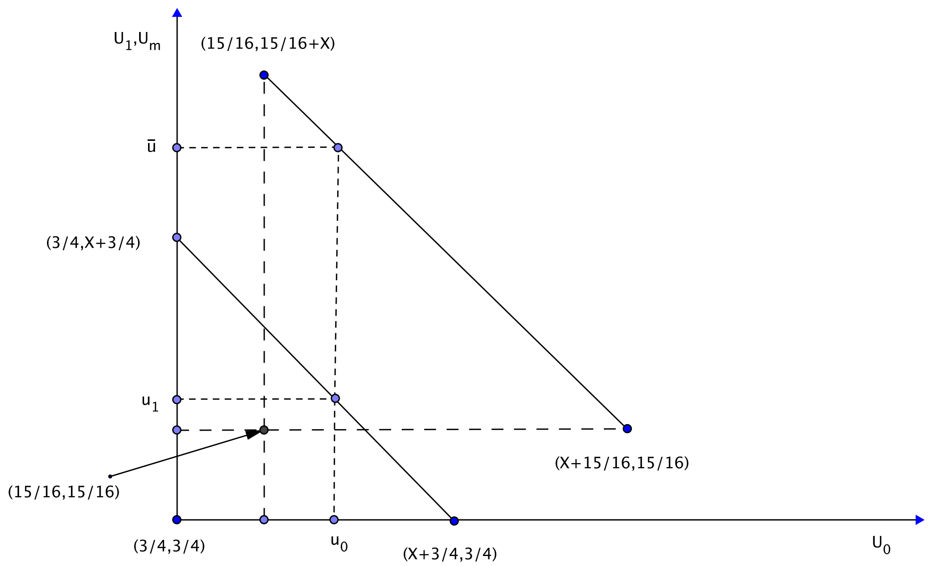


Figure 1:
If m's continuation payoff, u_m lies between $\bar{u}$ and u_1, then there exists a profitable deviation for 0 where they propose to m instead of 1, moving to the outer utility frontier and increasing their own continuation value from u_0.

Since $\partial u_i(y)/\partial y = 2(v_i - y)$, it follows that

$$\frac{\partial u_i(y)}{\partial y} + \frac{\partial u_j(y)}{\partial y} = 2(v_i - y) + 2(v_j - y) = 2(v_i + v_j - 2y) = 0 \Longleftrightarrow y = (v_i + v_j)/2$$

Case 1: Suppose the symmetric equilibrium is one where the two extreme players propose to one another, i.e. 1 proposes to 0 and 0 proposes to 1. We wil show that at the beginning of any subgame at which one of the extreme players is to propose to the other according to the equilibrium strategy profile, it would be profitable to propose to m instead. More precisely, there exists a proposal that offers some share of the resouces to m that player 0, say, would prefer to the one that offers a share to 1, and that this proposal would be accepted by m.

Let the total utility that can be "shared" between legislators i and j whenever i proposes to j be denoted by $U(i,j) = U_i(P_j^i) + U_j(P_j^i)$, where P_j^i is i's proposal to j.

We will show that such a deviation exists for player 0, but the symmetry of the setup implies that the argument is exactly the same for 1. Notice that whenever the proposals are interior, the proposed policy is located at the midpoint between the two legislators' ideal points. It's clear to see that since the ideal point of m is closer to 0 than is 1, $U(0,m) > U(0,1)$:

$$U(0,m) = U_0(P_m^0)+U_m(P_m^0) = 1-\left(0-\frac{1}{4}\right)^2+x_m^0X+1-\left(\frac{1}{2}-\frac{1}{4}\right)^2+\left(1-x_m^0\right)X = \frac{15}{8}+X$$

$$U(0,1) = U_0(P_1^0)+U_1(P_1^0) = 1-\left(0-\frac{1}{2}\right)^2+x_1^0X+1-\left(1-\frac{1}{2}\right)^2+\left(1-x_1^0\right)X = \frac{3}{2}+X$$

The argument can be most clearly seen by looking at figure 1. The continuation payoffs of m and 1 are on the y-axis, and 0's continuation payoff is on the x-axis. The utility frontier when 0 proposes to m is everywhere above that for the case when 0 proposes to 1 since $U(0,m) > U(0,1)$. If a strategy profile defines continuation payoffs u_0 and u_1 as depicted, then if u_m lies strictly below $\bar{u}$, a profitable deviation exists where 0 can propose some split of the resources and the policy $y_m^0 = 1/4$ that yield immediate utility for m greater than u_m. This means that m accepts this offer from 0. Moreover, by the definition of $\bar{u}$, this proposal would yield utility for 0 greater than u_0, their utility from proposing to 1. This follows by the observation that a horizontal line from any point on the y-axis below $\bar{u}$ intersects the outer utility frontier to the right of u_0.

To show that a profitable deviation for 0 exists, therefore, it suffices to show that $u_m < \bar{u}$. This can be shown by solving for u_m, u_1 and $\bar{u}$ from the five simultaneous equations (in five unknowns x^m, u_0, u_m, u_1, and x^0) below that derive from the binding incentive constraints associated with the strategy profile we specified. Notice that x^0 is proportion of resources kept by 0 when they propose to 1, and, by symmetry, it is also the proportion kept by 1 when they propose to 0, and $u(x) = 1 - x^2$.

1) 0 accepts m's offer:

$$(1 - x^m)X + u(1/4) = u_0$$

2) 1 accepts m's offer:

$$(1 - x^m)X + u(1/4) = u_1$$

3) 1 accepts 0's offer:

$$(1 - x^0)X + u(1/2) = u_1$$

4) m's continuation payoff:

$$u_m = \frac{\delta}{3}\left(0 + u(1/4) + x^m X + u(1/4) + 0 + u(1/4)\right)$$

5) 1's continuation payoff:

$$u_1 = \frac{\delta}{3}\left[(1 - x^0)X + u(1/2) + \frac{1}{2}(0 + u(3/4)) + \frac{1}{2}((1 - x^m)X + u(1/4)) + x^0 X + u(1/2)\right]$$

Once we solve for u_0, we can solve for $\bar{u}$ by the equality $U(0, m) = u_0 + \bar{u}$[12]. Since this exercise involves solving five simultaneous equations in five unknowns, we find the solutions numerically for some X and δ and verify that $u_m < \bar{u}$ holds.

Case 2: Now suppose the symmetric equilibrium is one in which the extreme players, 0 and 1, randomize between proposing to m and one another. Namely, they each propose to m with probability q, and with probability $(1 - q)$ they propose to the other extreme player. An equilibrium, therefore, is characterized by a vector

$$(x^m, q, x^0_m, x^0_1),$$

where m proposes $(1 - x^m)$ of the resources to each extreme player with probability 1/2, each extreme player proposes x^0_m of the resources to m with probability q and x^0_1 to the other extreme player with probability $(1 - q)$. We want to show that an extreme player can profitably deviate in the subgame in which they should propose to the other extreme player by proposing, instead, to m. In addition, we want to show that this proposal will be accepted by m. Similarly to the above case, we solve for u_m and u_1 from the equations characterizing the equilibrium. We have 6 unknowns: the 4 equilibrium quantities and 2 continuation payoffs (0 and 1's continuation payoffs are the same, so we only consider u_0 in the system below).

1) 0 accepts m's offer:

[12] This is clear to see from figure 1.

$$(1 - x^m)X + u(1/4) = u_0$$

2) 0 accepts 1's offer:

$$(1 - x_1^0)X + u(1/2) = u_1$$

3) m accepts 0 & 1's offers:

$$(1 - x_m^0)X + u(1/4) = u_m$$

4) m's continuation payoff:

$$u_m = \frac{\delta}{3}\left(q(1 - x_m^0)X + u(1/4) + x^m X + u(1/4) + q(1 - x_m^0)X + u(1/4)\right)$$

5) 1's continuation payoff:

$$\begin{aligned} u_1 =& \frac{\delta}{3}\Big[(1-q)[(1 - x_1^0)X + u(1/2)] + q[(1 - x_m^0)X + u(1/4)] \ldots \\ & \ldots + \frac{1}{2}(0 + u(3/4)) + \frac{1}{2}((1 - x^m)X + u(1/4)) + x^0 X + u(1/2)\Big] \end{aligned}$$

6) m's continuation payoff equals both extreme players' continuation payoffs, since only then can any one player be indifferent (and hence randomize) between proposing to m and proposing to the other extreme player.

$$u_m = u_0$$

The above system of equations has a solutions since it has 6 equations in 6 unknowns. From the value of u_0, we can find $\bar{u}$, and check to see if $\bar{u} < u_m$. Since this exercise involves solving 6 equations in 6 unknowns, we check it numerically. □

7.1.3 Proof of Claim 3

Proof. An equilibrium of this form is a vector

$$(q_m^0, x_m^0, q_1^0, p, x_0^m, x_1^m, q_m^1, x_m^1, x_0^1),$$

such that $q^0 = q^1 = 1$. An equilibrium, theferore can be characterized by 5 quantities:

$$x = (x_1, x_2, x_3, x_4, x_5) = (x_m^0, p, x_0^m, x_1^m, x_m^1)$$

At an equilibrium, it must be that the following list of conditions hold. Although these conditions are characterized by inequalities, in equilibrium, they must hold with equality since otherwise the proposer can increase the proportion they keep while still ensuring his proposal is accepted. This gives us 5 equation in 5 unknowns:

1) 0 accepts m's offer

$$f_1(x, v_m) = (1 - x_0^m)X + u\left(\frac{v_m}{2}\right) - u_0 = 0$$

2) 1 accepts m's offer

$$f_2(x, v_m) = (1 - x_1^m)X + u\left(\frac{1 - v_m}{2}\right) - u_1 = 0$$

3) m accepts 0's offer

$$f_3(x, v_m) = (1 - x_m^0)X + u\left(\frac{v_m}{2}\right) - u_m = 0$$

4) m accepts 1's offer

$$f_4(x, v_m) = (1 - x_m^1)X + u\left(\frac{1 - v_m}{2}\right) - u_m = 0$$

5) m is indifferent between proposing to 0 and 1

$$f_5(x, v_m) = x_0^m X + u\left(\frac{v_m}{2}\right) - x_1^m X - u\left(\frac{1 - v_m}{2}\right) = 0$$

u_i is i's continuation payoff:

$$u_0 = \frac{\delta}{3}\left[x_m^0 X + u\left(\frac{v_m}{2}\right) + p\left((1-x_0^m)X + u\left(\frac{v_m}{2}\right)\right) + (1-p)\left(0 + u\left(\frac{1+v_m}{2}\right)\right) + 0 + u\left(\frac{1+v_m}{2}\right)\right]$$

$$u_m = \frac{\delta}{3}\left[(1-x_m^0)X + u\left(\frac{v_m}{2}\right) + p\left(x_0^m X + u\left(\frac{v_m}{2}\right)\right) + (1-p)\left(x_1^m X + u\left(\frac{1-v_m}{2}\right)\right) + (1-x_m^1)X + u\left(\frac{1-v_m}{2}\right)\right]$$

$$u_1 = \frac{\delta}{3}\left[0 + u\left(\frac{2-v_m}{2}\right) + p\left(0 + u\left(\frac{2-v_m}{2}\right)\right) + (1-p)\left((1-x_1^m)X + u\left(\frac{1-v_m}{2}\right)\right) + x_m^1 X + u\left(\frac{1-v_m}{2}\right)\right]$$

Let $x = (x_m^0, p, x_0^m, x_1^m, x_m^1)$ be the unknowns of this system of equations, and v_m be the parameter. We know that a symmetric equilibrium exists, and so the system of equations can be solved at $(\bar{x}, 1/2)$. To show that there exists another v_m close to 1/2 at which an equilibrium exists, we need to show that we can locally solve these equations at $(\bar{x}, 1/2)$. By the implicit function theorem, this is equivalent to showing that the jacobian of this system with respect to x evaluated at $\bar{x}$ is non-singular:

$$Det(J) = \begin{vmatrix} \frac{\partial f_1(\bar{x},1/2)}{\partial x_1} & \dots & \frac{\partial f_1(\bar{x},1/2}{\partial x_5} \\ \vdots & & \vdots \\ \frac{\partial f_5(\bar{x},1/2)}{\partial x_1} & \dots & \frac{\partial f_5(\bar{x},1/2}{\partial x_5} \end{vmatrix} \neq 0$$

$$Det(J) = \begin{vmatrix} -\frac{\delta}{3}X & -\frac{\delta}{3}\left((1-x_0^m)X + u\left(\frac{v_m}{2}\right)\right) - u\left(\frac{1+v_m}{2}\right) & (p-1)X & 0 & 0 \\ 0 & \frac{-\delta}{3}\left(u\left(\frac{2-v_m}{2}\right) - (1-x_1^m)X - u\left(\frac{1-v_m}{2}\right)\right) & 0 & \left(\frac{\delta}{3}(1-p) - 1\right)X & \frac{-\delta}{3}X \\ X\left(\frac{\delta}{3} - 1\right) & -\frac{\delta}{3}\left[x_0^m X + u\left(\frac{v_m}{2}\right) - x_1^m X - u\left(\frac{1-v_m}{2}\right)\right] & -\frac{\delta}{3}pX & -\frac{\delta}{3}(1-p)X & \frac{\delta}{3}X \\ \frac{\delta}{3}X & -\frac{\delta}{3}\left[x_0^m X + u\left(\frac{v_m}{2}\right) - x_1^m X - u\left(\frac{1-v_m}{2}\right)\right] & -\frac{\delta}{3}pX & -\frac{\delta}{3}(1-p)X & X\left(\frac{\delta}{3} - 1\right) \\ 0 & 0 & X & -X & 0 \end{vmatrix}$$

We want to show that $Det(J) \neq 0$. Notice that J is a matrix of the following form:

$$\begin{matrix} a & b & c & 0 & 0 \\ 0 & d & 0 & e & f \\ g & h & i & j & k \\ l & m & n & o & p \\ 0 & 0 & q & r & 0 \end{matrix}$$

Moreover, for any matrices A ($n \times n$), B ($n \times m$) and C ($m \times m$), the following equality holds:

$$Det \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = Det(A)Det(D)$$

We want to transform J into this form. To do so for the general matrix above, we need to do the following operations that leave the determinant unchanged:

- multiply the first row by $-l/a$ and add it to the 4th
- multiply the second row by $\frac{1}{d}\left(\frac{l}{a}b - m\right)$ and add it to the 4th

These two operations yield a matrix of the form:

$$\begin{matrix} a & b & c & 0 & 0 \\ 0 & d & 0 & e & f \\ g & h & i & j & k \\ 0 & 0 & n - \frac{lc}{a} & o + \frac{e}{d}\left(\frac{l}{a}b - m\right) & p + \frac{f}{d}\left(\frac{l}{a}b - m\right) \\ 0 & 0 & q & r & 0 \end{matrix}$$

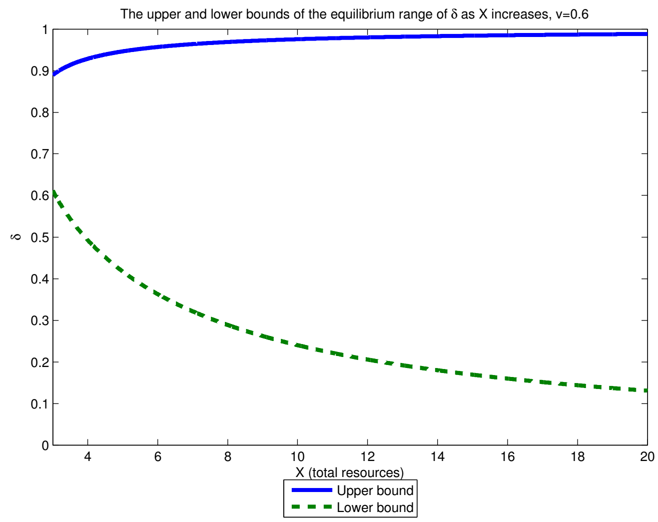
The determinant of this matrix is then simply the following:

$$\begin{aligned} Det(J) = Det & \begin{pmatrix} a & b & c \\ 0 & d & 0 \\ g & h & i \end{pmatrix} \times Det \begin{pmatrix} o + \frac{e}{d}\frac{l}{a}b - m) & p + \frac{f}{d}(\frac{l}{a}b - m) \\ r & 0 \end{pmatrix} \\ &= (adi + 0 + 0 - cdg - 0 - 0) \times -r\left[p + \frac{f}{d}\left(\frac{l}{a}b - m\right)\right] \\ &= (adi - cdg) \times r\left[\frac{f}{d}\left(m - \frac{l}{a}b\right) - p\right] \end{aligned}$$

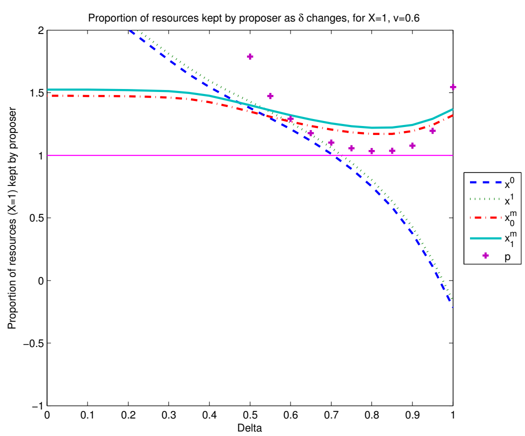
Whenever $\delta^2 p/9 \neq (p-1)(\delta - 3)/3$, the first term in the multiplication is not zero. Whether the second term is equal to zero or not is tedious to verify and, for the time being, we conjecture that it's not. The numerical exercises we conduct, the results of some of which are included in the appendix support this conjecture: that there exist equilibria of this form for $v_m \neq 1/2$. $\square$

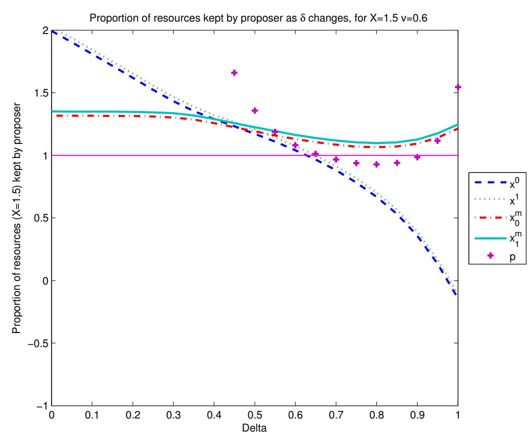
7.2 Numerical exercises

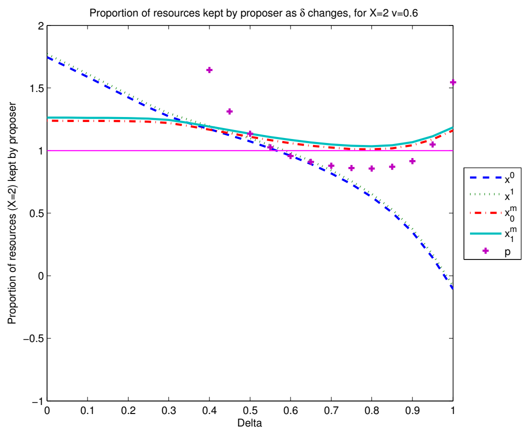
7.2.1 The range of δ for which an equilibrium exists (all x^i's $\in [0,1]$) as X increases, $v_m = 0.6$

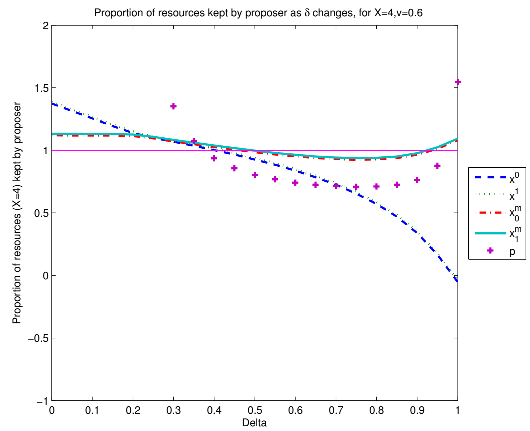


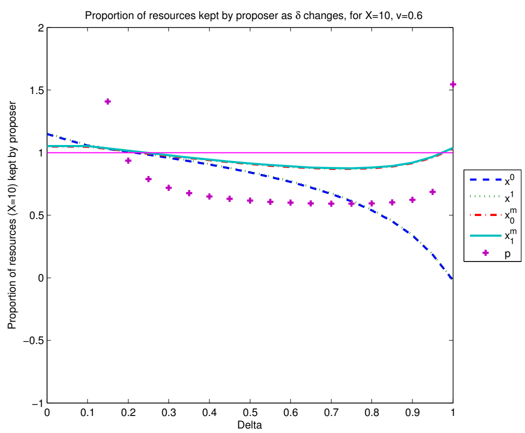
7.2.2 Equilibrium strategies for a fixed amount of resources, as δ changes with $v_m = 0.6$

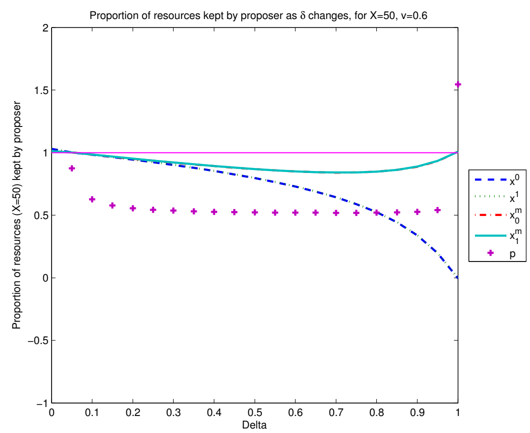




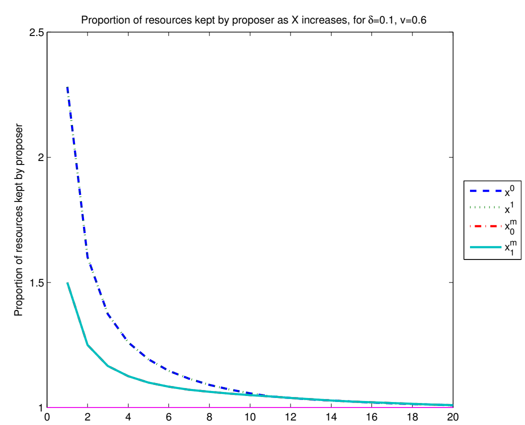


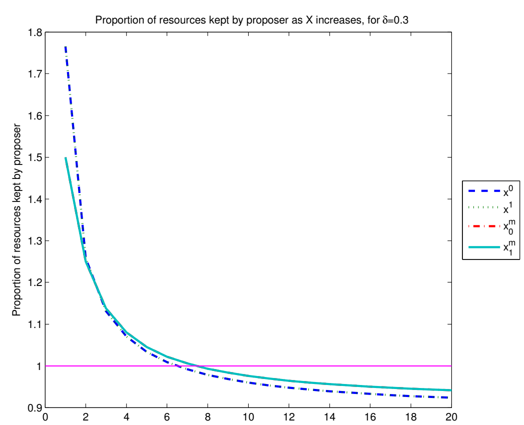


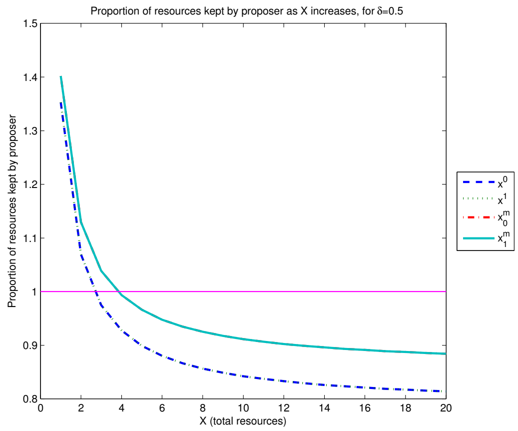


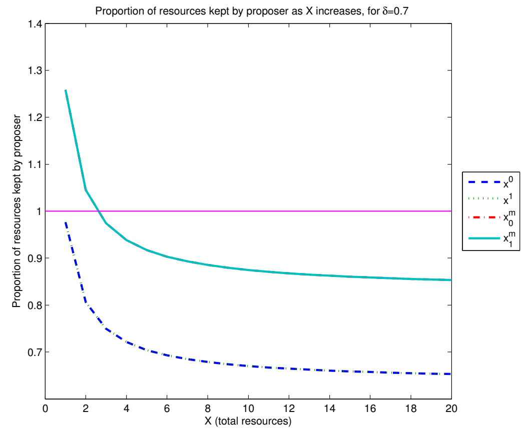


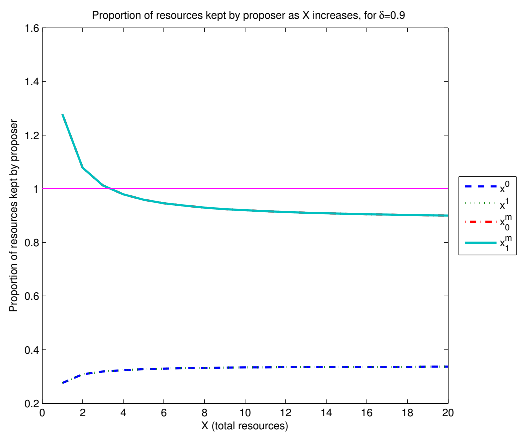
7.2.3 Equilibrium strategies for a fixed δ as the amount of resources, X, changes



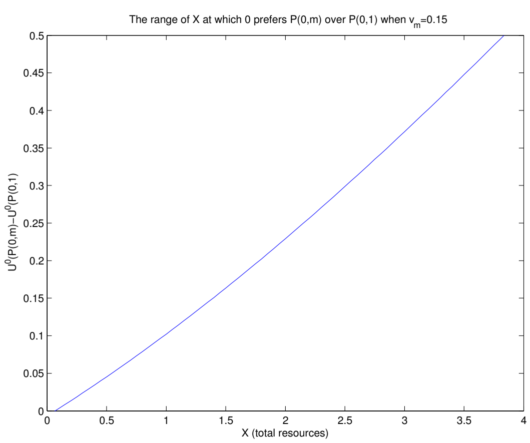


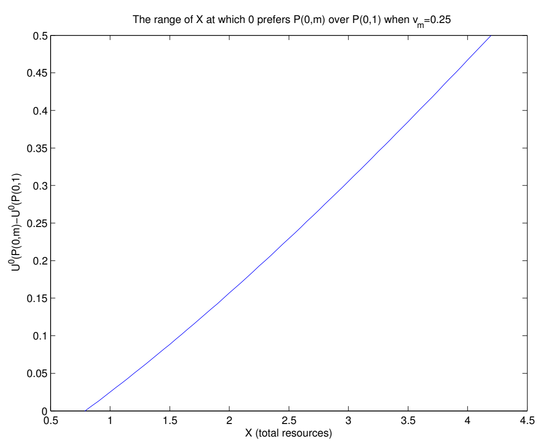


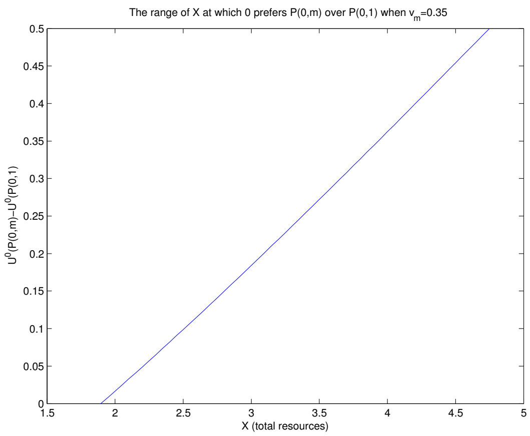


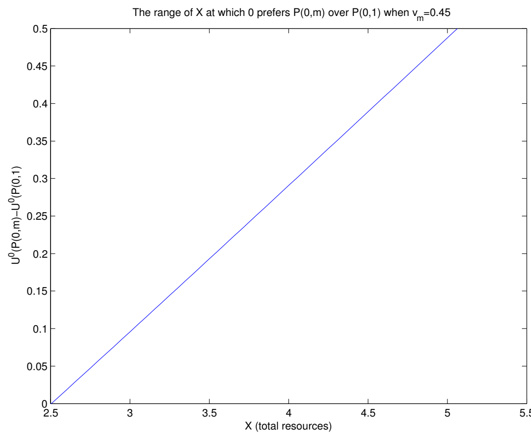


7.2.4 The range of X for which 0 prefers P(0,m) over P(0,1)






The range of X at which 0 prefers P(0,m) over P(0,1) when v_m=0.35
U^0(P(0,m))−U^0(P(0,1))
X (total resources)


The range of X at which 0 prefers P(0,m) over P(0,1) when v_m=0.45
U^0(P(0,m))−U^0(P(0,1))
X (total resources)